Module-3: Analytic Functions-II

1 Introduction

To make the C-R equations as sufficient, an additional condition on partial derivatives is essential and this is the condition of continuity.

Theorem 1. *(Sufficient Condition for Analyticity)*
A single valued continuous function $w = f(z) = u(x,y) + iv(x,y)$ is differentiable in a domain D if the four partial derivatives u_x, u_y, v_x, v_y exist, are continuous and satisfy C-R equations at each point of D.

Proof. *We have to show that $f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ exists at each point of D. Let $z = x+iy$ be any point of D. Since u_x, u_y, v_x, v_y exist and continuous at (x,y), $u(x,y)$ and $v(x,y)$ are differentiable at (x,y). Therefore,*

$$\begin{aligned} \Delta u &= u(x+\Delta x, y+\Delta y) - u(x,y) \\ &= u_x \Delta x + u_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \end{aligned}$$

where $\varepsilon_1, \ \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$, and

$$\begin{aligned} \Delta v &= v(x+\Delta x, y+\Delta y) - v(x,y) \\ &= v_x \Delta x + v_y \Delta y + \eta_1 \Delta x + \eta_2 \Delta y, \end{aligned}$$

where $\eta_1, \ \eta_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.
Since u and v satisfy C-R equations at (x,y), we have

$$\begin{aligned} \Delta w &= \Delta u + i\Delta v \\ &= u_x(\Delta x + i\Delta y) + v_x(i\Delta x - \Delta y) + (\varepsilon_1 + i\eta_1)\Delta x + (\varepsilon_2 + i\eta_2)\Delta y. \qquad (1) \end{aligned}$$

From (1) we get

$$\frac{\Delta w}{\Delta z} = u_x + iv_x + (\varepsilon_1 + i\eta_1)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\eta_2)\frac{\Delta y}{\Delta z}. \qquad (2)$$

Now

$$
\begin{aligned}
\mid (\varepsilon_1 + i\eta_1)\frac{\Delta x}{\Delta z} \mid &= \mid \varepsilon_1 + i\eta_1 \mid\mid \frac{\Delta x}{\Delta z} \mid \\
&\leq \mid \varepsilon_1 \mid + \mid \eta_1 \mid \to 0 \text{ as } (\Delta x, \Delta y) \to (0,0) \text{ i.e. as } \Delta z \to 0.
\end{aligned}
$$

Similarly we can get

$$\mid (\varepsilon_2 + i\eta_2)\frac{\Delta y}{\Delta z} \mid \to 0 \;\; as \;\; \Delta z \to 0.$$

Thus taking limit as $\Delta z \to 0$ we obtain from (2)

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x + iv_x.$$

Hence $f'(z)$ exist and is equal to $u_x + iv_x$. Since z is any point in D, $f(z)$ is differentiable in D. This proves the theorem.

Example 1. *Let $f = u + iv$ be analytic in a domain D. Show that f is constant in D if $f'(z) \equiv 0$ in D.*

Solution. *Since $f = u + iv$ is analytic in D, it is differentiable there and satisfies the C-R equations in D. Now $f'(z) = 0$ implies $u_x + iv_x = 0$. That is $u_x = 0$ and $v_x = 0$. So using C-R equations we have $u_y = 0$ and $v_y = 0$. Thus*

$$
\begin{aligned}
du &= u_x dx + u_y dy = 0 \\
i.e. \; u &= constant.
\end{aligned}
$$

Similarly we obtain $v = constant$ and so $f(z) = u + iv$ is constant.

Theorem 2. *Let $f(z) = u + iv$ be analytic in a domain D and $\mid f(z) \mid$ is equal to constant in D. Then $f(z)$ is constant in D.*

Proof. *Let $\mid f(z) \mid = c$, say. Then $u^2 + v^2 = c^2$. Differentiating with respect to x and y we get respectively*

$$uu_x + vv_x = 0, \tag{3}$$

$$uu_y + vv_y = 0. \tag{4}$$

Using C-R equations $u_x = v_y$ and $u_y = -v_x$, we obtain from (4) that

$$-uv_x + vu_x = 0. \tag{5}$$

From (3) and (5) we get $(u^2+v^2)u_x = 0$. *Now* $u^2+v^2=0$ *implies* $u=0=v$ *and hence* $f(z)=0$, *a constant function. If* $u^2+v^2 \neq 0$, *we have* $u_x = 0$. *Similarly from (3) and (5) we get* $v_x = 0$. *Hence* $u_x = v_x = u_y = v_y = 0$. *Thus*

$$\begin{aligned} du &= u_x dx + u_y dy = 0 \\ \text{i.e. } u &= \text{constant}. \end{aligned}$$

Similarly we obtain $v = \text{constant}$ *and so* $f(z) = u+iv$ *is constant.*

Theorem 3. *Let* $f = u+iv$ *be analytic in a domain* D. *Then* f *is constant in* D *if* $\arg f(z)$ *is constant in* D.

Proof. *Let* $\arg f(z) = c_1$, *a constant. Then* $v = cu$. *Differentiating with respect to* x *and* y *separately, we get*

$$v_x = cu_x, \quad v_y = cu_y.$$

This gives

$$f' = u_x + iv_x = (1+ic)u_x.$$

Again

$$\begin{aligned} f' &= v_y - iu_y = v_y - \frac{i}{c}v_y \\ &= (1-\frac{i}{c})v_y = (1-\frac{i}{c})u_x. \end{aligned}$$

If $u_x = 0$ *then* $f' = 0$, *which gives* f *is constant. Thus* $u_x \neq 0$. *Therefore,*

$$\begin{aligned} 1+ic &= 1-\frac{i}{c} \\ \text{i.e. } c^2 &= -1 \\ \text{i.e. } c &= \pm i. \end{aligned}$$

If $c = -i$, *then* $v_x = -iu_x$ *and* $v_y = -iu_y$. *Hence*

$$\begin{aligned} f' &= (1+ic)u_x = 2u_x \\ f' &= v_y - iu_y = -2iu_y, \end{aligned}$$

which is not possible. Therefore $c = i$ *and* $f' = (1+ic)u_x = 0$. *This shows that* f *is constant.*

Example 2. *Show that the function $f(z) = xy + iy$ is everywhere continuous but is not analytic.*

Solution. *Let $f(z) = u + iv$. Then $u = xy$ and $v = y$. Since u and v are continuous everywhere, $f(z)$ is continuous everywhere. Now,*

$$\frac{\partial u}{\partial x} = y, \; \frac{\partial u}{\partial y} = x, \; \frac{\partial v}{\partial x} = 0, \; \frac{\partial v}{\partial y} = 1.$$

Since

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x},$$

$f(z)$ is not an analytic function.

Example 3. *If $u = (x-1)^3 - 3xy^2 + 3y^2$, find v so that $u + iv$ is an analytic function of $x + iy$.*

Solution. *Here $\frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2$ and $\frac{\partial u}{\partial y} = -6xy + 6y$.*
By C-R equations we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy - 6y.$$

Integrating with respect to x we get

$$v = 3x^2y - 6xy + f(y). \tag{6}$$

This gives

$$\frac{\partial v}{\partial y} = 3x^2 - 6x + f'(y).$$

Also $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2$. Hence

$$\begin{aligned} 3x^2 - 6x + f'(y) &= 3(x-1)^2 - 3y^2 \\ \text{i.e. } f'(y) &= 3 - 3y^2. \end{aligned}$$

Integrating we get $f(y) = 3y - y^3 + c$, where c is a constant. Substituting this value of $f(y)$ in (6) we obtain

$$v = 3x^2y - 6xy + 3y - y^3 + c.$$

Example 4. *Given that the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D. Verify whether the functions $\overline{f(z)}$, $f(\overline{z})$, $\overline{f(\overline{z})}$ are analytic or not in D.*

Solution. *Since the function* $f(z) = u(x,y) + iv(x,y)$ *is analytic in a domain* D*, we have*

$$u_x = v_y \text{ and } u_y = -v_x$$

for all points of D*. Now*
$\overline{f(z)} = u(x,y) - iv(x,y)$ *will be analytic if*

$$u_x = (-v)_y \text{ and } u_y = -(-v)_x$$
$$\text{i.e. } u_x = -v_y \text{ and } u_y = v_x,$$

which is not the case. Hence $\overline{f(z)}$ *is not analytic in* D*.*
Again $f(\overline{z}) = u(x,-y) + iv(x,-y)$ *will be analytic if*

$$u_x = v_{-y} \text{ and } u_{-y} = -v_x$$
$$\text{i.e. } u_x = -v_y \text{ and } u_y = v_x,$$

which is not the case. Hence $f(\overline{z})$ *is not analytic in* D*.*
The function $\overline{f(\overline{z})} = u(x,-y) - iv(x,-y)$ *will be analytic in* D *if*

$$u_x = (-v)_{-y} \text{ and } u_{-y} = -(-v)_x$$
$$\text{i.e. } u_x = v_y \text{ and } u_y = -v_x.$$

Therefore, $\overline{f(\overline{z})}$ *is analytic in* D*.*

Example 5. *Show that the function* $f(z) = \frac{1}{z^4}$, $z \neq 0$ *is differentiable in indicated domain and find* $f'(z)$*.*

Solution. *Here we consider the polar system. The given function is*

$$f(z) = r^{-4}(\cos 4\theta - i\sin 4\theta).$$

Therefore,

$$u(r,\theta) = \frac{\cos 4\theta}{r^4}, \quad v(r,\theta) = -\frac{\sin 4\theta}{r^4}.$$

Thus

$$u_r = -\frac{4\cos 4\theta}{r^5}, \; u_\theta = -\frac{4\sin 4\theta}{r^4}, \; v_r = \frac{4\sin 4\theta}{r^5}, \; v_\theta = -\frac{4\cos 4\theta}{r^4}.$$

The first order partial derivatives of u and v, being rational continuous functions with non-vanishing denominators, are continuous. Also, the C-R equations

$$u_r = -\frac{4\cos 4\theta}{r^5} = \frac{1}{r}v_\theta, \text{ and } v_r = \frac{4\sin 4\theta}{r^5} = -\frac{1}{r}u_\theta$$

are satisfied. This concludes that $f'(z)$ exists and

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(-\frac{4\cos 4\theta}{r^5} + i\frac{4\sin 4\theta}{r^5}\right) \\ &= -\frac{4}{(re^{i\theta})^5} = -\frac{4}{z^5}. \end{aligned}$$